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Space-Energy Duality Generalized 4-Index Einstein Field Equation

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Abstract

The introduction of a new concept of space-energy duality serves to extend the applicability of the Einstein field equation in the context of a 4-index framework. The utilization of the Weyl tensor enables the derivation of Einstein's equations in the 4-index format. Additionally, a two-index field equation is presented, comprising a conventional Einstein field equation and a trace-free Einstein equation. Notably, the cosmological constant is associated with a novel concept that facilitates the encoding of space and energy information, thereby enabling the recognition of mutual interactions between space and energy in the presence of gravitational forces, as dictated by Einstein's field equations (EFE) and Trace-Free Einstein Equation (TFE).

Keywords

Einstein Field Equation, Trace-Free Einstein Equation, 4-Index, Cosmological Constant

1. Introduction

One of the most significant intellectual achievements of the 20th century is Einstein's general relativity (GR), which presents the current understanding of gravitation in modern physics. It provides a geometric explanation of gravity that does not conceptualize it as a traditional force, but rather as a result of the curvature of space-time. This curvature is directly linked to energy through Einstein's field equation [1]-[6].

Despite the successes achieved by Einstein's equations in general relativity, however, it is criticized for several failures in the field of formulating the total energy tensor, as the description of all the energy fields contributing to the studied system affecting space was not considered. Incorporates all potential substantial fields, including dark energy, but excludes the energy, momenta, or stresses related to the gravitational field itself because there isn't a correct energy stress tensor for the gravitational field.

For the theory of general relativity to be successful, it must fully describe the total energy tensor, so that it includes all possible prospects for sources of energy that contribute to the system. This accomplishment is only possible if we include inflation, dark matter, and dark energy in the energy stress tensor, all of which are believed to have major effects on the dynamics of the cosmos.

Since Einstein laid the general theory of relativity and the field equation, many attempts have been made to develop it [7] [8] [9]. The attempt to generalize Einstein's equation in the quadrilateral field led to the emergence of the Weyl tensor, which carries information about vacant space.

On the other hand, Einstein's tensor and energy tensor are scale-invariant. This scale-free functionality is further illustrated by the notion that gravitational radiation is captured by the Weyl curvature of space-time while the Ricci curvature provides compatibility with matter fields. Once a cosmological constant is recognized, however, the scaling feature is lost. Einstein thought that changing GR would result in a traceless theory since the violation of the scaling feature is so severe.

The fundamental concept entails the elimination of the trace component from Einstein's equation on both sides.

In a space-time characterized by n dimensions, when presented with a symmetric (0, 2) tensor X, we can establish its trace-free component as: $\hat{X}_{jl} = X_{jl} - \frac{1}{n} g_{jl} g^{jl} \left(X_{jl} \right)$. The idea then is to replace Einstein's equation with the trace free part on both sides: $\hat{G}_{jl} = \hat{T}_{jl}$. All vacuum solutions with some cosmological constant solve this equation.

2. Communication between Space and Energy

The theory of relativity has been instrumental in exploring the concept of communication between space and energy. By establishing a mathematical framework, this theory has provided insights into how space reacts to the movement of matter within it. It suggests that space can be curved in response to matter, while matter, in turn, influences the curvature of space. However, the underlying mechanism that governs this interaction remains a subject of inquiry. The question arises: How does matter transmit information to space, prompting a response from it?

To address these inquiries, a model has been developed to shed light on the process of information transfer between two systems with distinct physical properties. This model postulates the necessity of an intermediary to facilitate the transfer of information. The proposed process unfolds as follows: when energy interacts with space, the inertial space detects the presence of energy, such as an inertial mass. In response, it generates a gravitational mass, also known as vacuum energy, which encapsulates all the relevant information about the vacant space. This vacuum energy is then transmitted to the inertial mass. Similarly, the inertial mass creates a virtual space, referred to as gravitational space or vacuum space, which carries all the pertinent information about the inertial mass. This mutual exchange of information between space and energy can be conceptualized as a form of duality, termed space-energy duality.

The interaction between space and energy is governed by an equation that incorporates certain terms. These terms are represented by tensors, which are capable of conveying information about vacuum space and vacuum energy. These tensors, such as the Weyl tensor, vanish out beyond the confines of the described space to rectify and generalize Einstein's field equation.

The initial state of space when it interacts with external energy can be defined as a vacuum. It is important to note that space does not necessarily have to be devoid of energy or mass. The concept of flatness, which refers to the relative state of space, is contingent upon the perspective of the observer.

This idea can be compared to what is presented by Wheeler-Feynman's theory of radiation. John Cramer called the Wheeler-Feynman theory of radiation, the transactional interpretation of quantum mechanics, where sub-atomic particles produce (offer) and (confirmation) waves in time and anti-time respectively, so that in effect an action in anti-time provides the electron pair with advanced information about how they will interact. And although the electrons may seem to "consider" all possible interactions, the electrons, therefore, end up interacting only in what is one path for each interaction, and the range of quantum probabilities for a set of interactions [10].

3. The Model of Space-Energy Duality

3.1. Formulation of the Dual Space-Energy

The characterization of the Ricci tensor is based on the measurement of shape deformation along geodesics in space within the framework of general relativity, which is established in the pseudo-Riemannian setting. This characterization is evident through the inclusion of the Ricci tensor in the Raychaudhuri equation. Consequently, the Einstein field equations propose that the pseudo-Riemannian metric can effectively describe space-time, with a remarkably straightforward relationship between the Ricci tensor and the matter content of the universe.

Einstein expressed the equations of general relativity using 2-index tensors, $R_{ik} - \frac{1}{2} R g_{ik} = T_{ik}$ with R_{ik} representing the Ricci curvature tensor and its scalar R, g_{ik} denoting the metric tensor, and T_{ik} representing the energy momentum tensor. This particular formulation disregarded the influence of vacuum, as it was encoded in the Wyle tensor, which becomes zero in the 2-index form.

The comprehensive description of space-time curvature is provided by the Riemann curvature tensor, which is a rank 4 tensor. If the Riemann curvature tensor is uniformly zero in all regions of space-time, it signifies the flatness of our space-time. In the scenario of an empty space that is flat, this implies that R_{ijkl} equals zero. When pure energy, represented as T_{ijkl}, interacts with the aforementioned space, an interaction between the space and energy takes place, leading to the emergence of interaction terms. These interaction terms are commonly known as pseudo tensors.

The Weyl conformal tensor C_{ijkl} and the components of the Riemann curvature tensor of general relativity R_{ijkl}, that solely involve the Ricci tensor R_{jl} and the curvature scalar R are the constituents of the Riemann curvature tensor. The Weyl tensor has the property of vanishing upon contraction, $g^{ik}C_{ijkl} = 0$, which implies that the information it carries regarding the gravitational field in vacuum is absent from the well-known Einstein equation.

From the aforementioned, a comprehensive equation can be formulated which encompasses the initial boundaries of the system and the terms of interaction in the shape of an amalgamated equation,

$$R_{ijkl} + \hat{R}_{ijkl} = T_{ijkl} + \hat{T}_{ijkl} + \mathcal{O}_{ijkl} \tag{1}$$

The total vacuum tensors are represented by $\hat{T}_{ijkl}$ and $\hat{R}_{ijkl}$, while the interaction term is denoted as $\mathcal{O}_{ijkl}$.

The Einstein field equation can be derived by contracting Equation (1) in two indices. This derivation involves the utilization of total vacuum tensors, namely $\hat{T}_{ijkl}$ and $\hat{R}_{ijkl}$, as well as the interaction term $\mathcal{O}_{ijkl}$. The main goal is to construct a collection of fourth-order tensors, denoted as $\tilde{R}_{jl}, \tilde{T}_{jl}, \tilde{R}$, and $\tilde{T}$, in order to determine the vacuum fields. These tensors should possess the same symmetries as the Riemann tensor and incorporate the metric tensors.

To achieve this objective, two sets of fourth-order tensors will be utilized, which can be defined as follows,

$$g_{ijkl}\tilde{T} = g_{ik}g_{jl}\tilde{T} - g_{il}g_{jk}\tilde{T} \tag{2}$$

$$g_{ijkl}\tilde{R} = g_{ik}g_{jl}\tilde{R} - g_{il}g_{jk}\tilde{R} \tag{3}$$

$$g_{ik}\tilde{T}_{jl} - g_{jk}\tilde{T}_{il} + g_{jl}\tilde{T}_{ik} - g_{il}\tilde{T}_{jk} = g_{ijk\rho}\tilde{T}_l^{\rho} + g_{ij\rho l}\tilde{T}_k^{\rho} \tag{4}$$

$$g_{ik}\tilde{R}_{jl} - g_{jk}\tilde{R}_{il} + g_{jl}\tilde{R}_{ik} - g_{il}\tilde{R}_{jk} = g_{ijk\rho}\tilde{R}_l^{\rho} + g_{ij\rho l}\tilde{R}_k^{\rho} \tag{5}$$

The equivalence between the combination of Equations (2)-(5) and the Riemannian tensor and energy-momentum tensor signifies that by using Equations (2)-(5), we can formulate the Riemannian and energy-momentum tensors for vacuum, see Frédéric Moulin [9] and references there in,

$$\hat{R}_{ijkl} = g_{ijkl}\tilde{R} + g_{ijk\rho}\tilde{R}_l^{\rho} + g_{ij\rho l}\tilde{R}_k^{\rho} + \mathcal{O}\left(\tilde{R}\right) + \tilde{R}_{ijkl} \tag{6}$$

$$\hat{T}_{ijkl} = g_{ijkl}\tilde{T} + g_{ijk\rho}\tilde{T}_l^{\rho} + g_{ij\rho l}\tilde{T}_k^{\rho} + \mathcal{O}\left(\tilde{T}\right) + \tilde{T}_{ijkl} \tag{7}$$

In order to derive the comprehensive expression for Equation (1) as the 4-index Einstein equation of general relativity, the substitution of the Riemannian and energy-momentum tensors for vacuum in a linear is required.

$$\begin{aligned}&\left(R_{ijkl} + \alpha\tilde{R}_{ijkl}\right) - \left(\beta\tilde{T}_{ijkl} + T_{ijkl}\right) + \Big\{\alpha_1 g_{ijkl}\tilde{R} - \beta_1 g_{ijkl}\tilde{T} + \alpha_2\left(g_{ijk\rho}\tilde{R}_l^{\rho} + g_{ij\rho l}\tilde{R}_k^{\rho}\right)\\&-\beta_2\left(g_{ijk\rho}\tilde{T}_l^{\rho} + g_{ij\rho l}\tilde{T}_k^{\rho}\right)\Big\} = \mathcal{O}_{ijkl}\left(\tilde{R},\tilde{T}\right)\end{aligned} \tag{8}$$

where $\alpha, \alpha_1, \alpha_2$ are three arbitrary parameters that can be determined.

$\mathcal{O}_{ijkl}\left(\tilde{R}, \tilde{T}\right)$ the dual coupling tensor (interaction term).

3.2. Four-Index Einstein Equation

By substituting the values of the parameters α, α_1, and α_2 (as provided in **Appendix 1**) into Equation (8), it becomes possible to express a generalized field equation in a simplified form that encompasses only a single parameter;

$$\begin{aligned} & R_{ijkl} + \alpha \tilde{R}_{ijkl} + \frac{1-\alpha}{n-2}\left(g_{ijk\rho}\tilde{R}_l^{\rho} + g_{ij\rho l}\tilde{R}_k^{\rho}\right) - \frac{1-\alpha}{(n-1)(n-2)} g_{ijkl}\tilde{R} \\ & = \left\{ T_{ijkl} + \beta \tilde{T}_{ijkl} + \frac{1-\beta}{n-2}\left(g_{ijk\rho}\tilde{T}_l^{\rho} + g_{ij\rho l}\tilde{T}_k^{\rho}\right) - \frac{1-\beta}{(n-1)(n-2)} g_{ijkl}\tilde{T} \right\} + \mathcal{O}_{ijkl}\left(\tilde{R}, \tilde{T}\right) \end{aligned} \tag{9}$$

The definition of the interaction tensor can be established by considering $\mathcal{O}_{ijkl}\left(\tilde{R}, \tilde{T}\right)$ as

$$\mathcal{O}_{ijkl}\left(\tilde{R}, \tilde{T}\right) = -\frac{1}{2}\left(\frac{\beta_1}{\beta_2} g_{ijkl}\tilde{T} - \frac{\alpha_1}{\alpha_2} g_{ijkl}\tilde{R}\right) = \frac{1}{2(n-1)}\left(g_{ijkl}\tilde{R} - g_{ijkl}\tilde{T}\right) \tag{10}$$

Upon substituting Equation (10) into Equation (9), the resulting equation can be identified as the 4-index Einstein equation.

$$\begin{aligned} & R_{ijkl} + \alpha \tilde{R}_{ijkl} + \left\{ \frac{1-\alpha}{n-2}\left(g_{ijk\rho}\tilde{R}_l^{\rho} + g_{ij\rho l}\tilde{R}_k^{\rho}\right) - \frac{n-2\alpha}{2(n-1)(n-2)} g_{ijkl}\tilde{R} \right\} \\ & = T_{ijkl} + \beta \tilde{T}_{ijkl} + \left\{ \frac{1-\beta}{n-2}\left(g_{ijk\rho}\tilde{T}_l^{\rho} + g_{ij\rho l}\tilde{T}_k^{\rho}\right) - \frac{n-2\beta}{2(n-1)(n-2)} g_{ijkl}\tilde{T} \right\} \end{aligned} \tag{11}$$

Equation (11) presents a generalization of the 4-index Einstein equation, encompassing a wider range of phenomena. The vacuum components within Equation (11) bear a similarity to those obtained through the application of the principle of least action by Frédéric Moulin [9]. By conducting the tonsorial contraction (**Appendix 2**) of this equation, Einstein's equation of general relativity is derived, regardless of the particular values assigned to α, β, and n.

The total vacuum tensors, $\hat{R}_{ijkl}$ and $\hat{T}_{ijkl}$ can be defined as

$$\hat{R}_{ijkl} = \alpha \tilde{R}_{ijkl} + \left\{ \frac{1-\alpha}{n-2}\left(g_{ijk\rho}\tilde{R}_l^{\rho} + g_{ij\rho l}\tilde{R}_k^{\rho}\right) - \frac{n-2\alpha}{2(n-1)(n-2)} g_{ijkl}\tilde{R} \right\} \tag{12}$$

$$\hat{T}_{ijkl} = \beta \tilde{T}_{ijkl} + \left\{ \frac{1-\beta}{n-2}\left(g_{ijk\rho}\tilde{T}_l^{\rho} + g_{ij\rho l}\tilde{T}_k^{\rho}\right) - \frac{n-2\beta}{2(n-1)(n-2)} g_{ijkl}\tilde{T} \right\} \tag{13}$$

If the space is empty, $T_{ijkl}, \hat{R}_{ijkl}$ and their components vanish, and Equation (11) reads,

$$R_{ijkl} = \beta \tilde{T}_{ijkl} + \left\{ \frac{1-\beta}{n-2}\left(g_{ijk\rho}\tilde{T}_l^{\rho} + g_{ij\rho l}\tilde{T}_k^{\rho}\right) - \frac{n-2\beta}{2(n-1)(n-2)} g_{ijkl}\tilde{T} \right\} = \hat{T}_{ijkl} \tag{14}$$

Equation (14) indicates that space has the ability to generate dark energy through its interaction process, which can effectively interact with external energy and matter. This interaction displays a duality similar to that of matter.

Contraction of (14) gives

$$R_{jl} = \tilde{T}_{jl} - \frac{1}{2} g_{jl} \tilde{T}, \quad \tilde{T} = -\frac{2}{n-2} R \quad \text{and} \quad R_{jl} - \frac{1}{n-2} g_{jl} R = \tilde{T}_{jl} \tag{15}$$

Equation (15) shows the deformation of vacant space consumes its internal energy (vacuum energy).

If the energy does not interact with space, R_{ijkl}, $\hat{T}_{ijkl}$ and their components have vanished, and Equation (11) reads,

$$T_{ijkl} = \alpha \tilde{R}_{ijkl} + \left\{ \frac{1-\alpha}{n-2} \left(g_{ijk\rho} \tilde{R}_l^{\rho} + g_{ij\rho l} \tilde{R}_k^{\rho} \right) - \frac{n-2\alpha}{2(n-1)(n-2)} g_{ijkl} \tilde{R} \right\} = \hat{R}_{ijkl} \tag{16}$$

Energy or matter can generate a distinct space, known as dark space or interaction space, which operates based on its interaction with any given space. This behavior exhibits a resemblance to the dual nature of matter.

$$T_{jl} = \tilde{R}_{jl} - \frac{1}{2} g_{jl} \tilde{R}, \quad \tilde{R} = -\frac{2}{n-2} T \quad \text{and} \quad T_{jl} - \frac{1}{n-2} g_{jl} T = \tilde{R}_{jl} \tag{17}$$

Equation (17) elucidates that the transmission of information regarding the intended curvature is facilitated by the matter. The generation of this spatial configuration, however, necessitates the utilization of energy resources. Therefore, Equations (15) and (17) represent two aspects of the dual energy space process.

Sakharov conjectures that space-time curvature is determined by the distribution of vacuum energy, and Equation (17) shows the deformation of pure energy annihilates its vacuum space [11].

Colella, Overhauser, and Werner [12], demonstrated in 1975 that de Broglie waves are influenced similarly by gravitational potentials. That experiment measured the gravitational phase shift of neutron waves.

Wheeler has called attention to a proposal by Sakharov, that gravitation ultimately arises from variation in the quantum zero-point energy of the vacuum [13].

3.3. Weyl Tensor

The Weyl tensor measures the curvature of space-time or a pseudo-Riemannian manifold and represents the tidal force experienced by a body moving along a geodesic. Unlike the Riemann curvature tensor, the Weyl tensor does not indicate changes in the volume of the body, but rather only the distortion of its shape due to tidal forces. The Ricci curvature, or trace component of the Riemann tensor, provides information on volume changes caused by tidal forces, while the Weyl tensor is the traceless component of the Riemann tensor.

To find the Weyl tensor, first, we define the tensor C_{ijkl} as

$$C_{ijkl} = \left\{ \frac{1}{n-2} \left(g_{ijk\rho} \tilde{R}_l^{\rho} + g_{ij\rho l} \tilde{R}_k^{\rho} \right) - \frac{1}{(n-1)(n-2)} g_{ijkl} \tilde{R} \right\} \tag{18}$$

We used

$$g^{ik}\left(g_{ijk\rho}\tilde{R}_l^{\rho}+g_{ij\rho l}\tilde{R}_k^{\rho}\right)=(n-2)\tilde{R}_{jl}+g_{jl}\tilde{R}$$

The tensorial contraction of (18) in two indices, gives

$$C_{jl}=\tilde{R}_{jl} \tag{19}$$

Then the Weyl tensor can be written as

$$\mathcal{C}_{ijkl}=\tilde{R}_{ijkl}-\left\{\frac{1}{n-2}\left(g_{ijk\rho}\tilde{R}_l^{\rho}+g_{ij\rho l}\tilde{R}_k^{\rho}\right)-\frac{1}{(n-1)(n-2)}g_{ijkl}\tilde{R}\right\} \tag{20}$$

Where the contraction of (20) is vanishing as

$$g^{ik}\mathcal{C}_{ijkl}=0 \tag{21}$$

3.4. Einstein Tensor

To find Einstein's tensor we define the tensor G_{ijkl} as 4-index tensor,

$$G_{ijkl}=\left\{\frac{1}{n-2}\left(g_{ijk\rho}\tilde{R}_l^{\rho}+g_{ij\rho l}\tilde{R}_k^{\rho}\right)-\frac{n}{2(n-1)(n-2)}g_{ijkl}\tilde{R}\right\} \tag{22}$$

We use, $g^{ik}\left(g_{ijk\rho}\tilde{R}_l^{\rho}+g_{ij\rho l}\tilde{R}_k^{\rho}\right)=(n-2)\tilde{R}_{jl}+g_{jl}\tilde{R}$ to contract Equation (22) as (**Appendix 2** (A2.5))

$$G_{jl}=\tilde{R}_{jl}-\frac{1}{2}g_{jl}\tilde{R} \tag{23}$$

Now, the representation of the 4-index Einstein tensor is depicted on the left-hand side of Equation (11) as follows:

$$E_{ijkl}=R_{ijkl}+\left\{G_{ijkl}+\alpha C_{ijkl}\right\} \tag{24}$$

The contraction of Equation (24) yields the two-index Einstein tensor;

$$E_{jl}=\bar{R}_{jl}-\frac{1}{2}g_{jl}\tilde{R} \tag{25}$$

The effective Ricci curvature tensor $\bar{R}_{jl}$ is defined as

$$\bar{R}_{jl}=\left(R_{jl}+\tilde{R}_{jl}\right) \tag{26}$$

3.5. 4-Index Energy-Momentum Tensor

Furthermore, it is possible to ascertain the energy tensor by dividing it into two distinct tensors. The first tensor pertains to the vacuum, whereas the second one is linked to the external energy. It is imperative to note that the first tensor must disappear when two indices are contracted, specifically $\tilde{\mathcal{T}}_{jl}=0$.

Conversely, the second tensor does not possess this characteristic and remains non-zero, denoted as $\mathcal{T}_{jl}\neq 0$.

The first tensor, known as the gravitational energy tensor or vacuum energy tensor, is now being defined as,

$$\tilde{\mathcal{T}}_{ijkl}=\tilde{T}_{ijkl}-\left\{\frac{1}{n-2}\left(g_{ijk\rho}\tilde{T}_l^{\rho}+g_{ij\rho l}\tilde{T}_k^{\rho}\right)-\frac{1}{(n-1)(n-2)}g_{ijkl}\tilde{T}\right\} \tag{27}$$

The contraction of the gravitational energy momentum tensor $\tilde{\mathcal{T}}_{ijkl}$ gives,

$$\tilde{\mathcal{T}}_{jl} = 0 \tag{28}$$

The traceless property of the gravitational energy momentum tensor $\tilde{\mathcal{T}}_{ijkl}$ signifies that it does not possess any trace (traceless).

The inertial (non-gravitational) energy tensor or external energy tensor can be defined as

$$\mathcal{T}_{ijkl} = \left\{ \frac{1}{n-2}\left(g_{ijk\rho}\tilde{T}_l^{\rho} + g_{ij\rho l}\tilde{T}_k^{\rho} \right) - \frac{n}{2(n-1)(n-2)} g_{ijkl}\tilde{T} \right\} \tag{29}$$

Contraction of Equation (29) gives,

$$\mathcal{T}_{jl} = \tilde{T}_{jl} - \frac{1}{2} g_{jl}\tilde{T} \tag{30}$$

To determine the complete energy-momentum tensor, it is necessary to combine the gravitational energy-momentum tensor and the non-gravitational energy-momentum tensor in a linear manner as

$$T_{ijkl}^{sm} = T_{ijkl} + \left\{ \mathcal{T}_{ijkl} + \beta\tilde{\mathcal{T}}_{ijkl} \right\} \tag{31}$$

We contract (31) in two indices to obtain the two-index total energy-momentum tensor, using the same method as before,

$$T_{jl}^{sm} = \bar{T}_{jl} - \frac{1}{2} g_{jl}\tilde{T} \tag{32}$$

The effective energy-momentum tensor $\bar{T}_{jl}$, which comprises the sum of $\tilde{T}_{jl}$ and T_{jl}, represents both the gravitational and non-gravitational effects. It characterizes the actual energy involved in the interaction process within the system, as described by the energy-momentum tensors.

3.6. Generalized Field Equation

To arrive at a generalized formula for the Einstein field equation, based on the previous equations of the 4-index Einstein tensor and the complete energy-momentum tensor,

$$E_{ijkl} = T_{ijkl}^{sm} \tag{33}$$

Upon contraction of Equation (33), the resulting outcome is the two-index field equation,

$$\bar{R}_{jl} - \frac{1}{2} g_{jl}\tilde{R} = \bar{T}_{jl} - \frac{1}{2} g_{jl}\tilde{T} \tag{34}$$

Equation (34) can be restated by incorporating the terms that depict the reciprocal impact of space and energy, manifested as vacuum terms.

$$R_{jl} - g_{jl}\Lambda = T_{jl} \tag{35}$$

It can be noted that all the information that space and energy need to recognize each other is encoded in the term $g_{jl}\Lambda$. Therefore, we find that $g_{jl}\Lambda$ controls the final form of Einstein's field equation.

There are two contexts in Equation (35). The first one if the deferential $\nabla^{l}\left(R_{jl}-g_{jl}\Lambda\right)=0$, Equation (35) construes to Einstein field equation. The second is $g^{jl}\left(R_{jl}-g_{jl}\Lambda-T_{jl}\right)=0$, in this case, Equation (35) construes to Trace-free Einstein field equation.

Using Equation (34) and Equation (35), Λ can be defined as follows

$$\Lambda=\frac{n-2}{2n}\mathbb{R} \tag{36}$$

The equation's interaction effect is represented by the scalar $\mathbb{R}$, which is the result of combining the duality of space and energy in a scalar summation

$$\mathbb{R}=\left\{\tilde{R}-\tilde{T}\right\}=\frac{2n}{n-2}\Lambda \tag{37}$$

In Equation (37) if the effective vacuum scalar curvature is $\mathbb{R}=\left\{\tilde{R}-\tilde{T}\right\}$ and effective cosmological constant Λ, we find the solution of quadratic gravity of field equation.

The Ricci curvature from (37) can be define as

$$\mathbb{R}_{jl}=\frac{2}{n-2}g_{jl}\Lambda \tag{38}$$

Equation (38) is the exact solution of quadratic gravity [14] [15]. In 4 dimensions ($n = 4$) Equations (36)-(38) construe to vacuum Einstein equations, which confirms the validity of our assumption that $\left\{\tilde{R},\tilde{T}\right\}$ are vacuum parts.

4. Einstein Field Equation and the Cosmological Constant

4.1. Return to Einstein's Original Equations

To reduce Equation (40) to the Einstein field equation, we put

$$R=2\Lambda=\frac{n-2}{n}\left\{\tilde{R}-\tilde{T}\right\} \quad \text{or} \quad T=-2\Lambda=\frac{n-2}{n}\left\{\tilde{T}-\tilde{R}\right\} \tag{39}$$

$$R_{jl}=\frac{n-2}{n^{2}}g_{jl}\left\{\tilde{R}-\tilde{T}\right\} \quad T_{jl}=\frac{n-2}{n^{2}}g_{jl}\left\{\tilde{T}-\tilde{R}\right\}$$

We use $R_{jl}=\frac{1}{n}g_{jl}g^{jl}\left(R_{jl}\right)=\frac{1}{n}g_{jl}\left(g^{jl}R_{jl}\right)=\frac{1}{n}g_{jl}R$, $g^{jl}R_{jl}=R$

The Schur lemma asserts that when the Ricci tensor is a multiple of the metric at each point [16] [17] [18], the metric is necessarily Einstein, except in the case of two dimensions. Furthermore, it implies that, apart from two dimensions, a metric is Einstein if and only if there exists a relationship between the Ricci tensor and the scalar curvature denoted by $R_{jl}=\frac{1}{n}g_{jl}R$. Equation (39) represents the scalar curvature of the Lagrangian density associated with the Einstein-Hilbert action. where, $S=\frac{1}{2}\int\left(R-2\Lambda\right)\sqrt{-g}\,\mathrm{d}^{4}x$.

The Euler–Lagrange equations for this Lagrangian under variations in the metric constitute the vacuum Einstein field equations with cosmological constant $R_{ik}-\frac{1}{2}Rg_{ik}+\Lambda g_{ik}=0$. Because the interaction term in vacuum remains constant, $\left\{\tilde{R}-\tilde{T}\right\}$ is a constant (see **Appendix 3**).

It is evident from (39) that the impact of gravity is contingent upon two factors: the geometric characteristics, elucidated by Einstein's interpretation, and the energy (mass), expounded by quantum mechanics. The weak equivalence principle of general relativity posits that the trajectory of a test particle in a gravitational field remains unaffected by its mass. Conversely, within the realm of quantum mechanics, the movement of a particle is intricately intertwined with its mass.

4.2. Trace-Free Einstein's Field Equation

The Trace-Free of Einstein's field Equation (TFE) can be obtained by defining, as indicated in Equations (35) and (39),

$$R=\frac{n-2}{2}\tilde{R} \quad \text{and} \quad T=\frac{n-2}{2}\tilde{T}, \tag{40}$$

Equation (34) corresponds to the formulation of the Trace-Free Einstein Equations (TFE) in a manifold of n dimensions.

$$\hat{G}_{jl}=\hat{T}_{jl} \tag{41}$$

The trace free Einstein tensor is $\hat{G}_{jl}=R_{jl}-\frac{1}{n}g_{jl}R$ and the trace free energy momentum tensor is $\hat{T}_{jl}=T_{jl}-\frac{1}{n}g_{jl}T$.

It is essential that the symmetry on both sides of the equation match. These become the gravitational field equations that we use. Suppose (l, g) represents a solution to the vacuum version of (41). In this scenario, it is implied that the traceless component of the Einstein tensor, and consequently the trace free portion of the Ricci tensor, completely disappears. Specifically, this implies that the Ricci tensor must be directly proportional to the metric g. When the dimension of space-time is at least 3, the aforementioned proportionality indicates that (l, g) is Einstein. Consequently, in the case of a vacuum, the function $\hat{G}_{jl}=0$ is equivalent to $G_{jl}+\Lambda g_{jl}$ for a certain unspecified constant Λ.

The traces of Equation (41) give

$$g^{jl}\left(R_{jl}-\frac{1}{n}g_{jl}R\right)=0 \quad \text{and} \quad g^{jl}\left(T_{jl}-\frac{1}{n}g_{jl}T\right)=0 \tag{42}$$

4.3. The Cosmological Constant

In 4 space-time dimensions, Einstein demonstrated that the Einstein-Maxwell theory adheres to the same principle, whereby any solution of the Einstein-Maxwell version of (41) is also a solution of the standard Einstein-Maxwell equation with cosmological constant. This is due to the conformal nature of the Maxwell field in 4 dimensions.

The energy-momentum tensor T_{jl}'s conservation is now an independent assumption and not a byproduct of the geometrical identity [19].

$$\nabla^{l}\left(R_{jl}-\frac{1}{n}g_{jl}R\right)=\nabla^{l}\left(T_{jl}-\frac{1}{n}g_{jl}T\right) \tag{43}$$

This differentiation interprets to the relation,

$$\nabla^{a}R=-\nabla^{a}T \tag{44}$$

By integration (44),

$$R+T=n\lambda \tag{45}$$

where λ is integration constant.

Substituting Equation (45) in (41) to eliminate T,

$$R_{jl}-\frac{2}{n}g_{jl}R+g_{jl}\lambda=T_{jl} \tag{46}$$

In the realm of four dimensions, Equation (46) can be expressed as,

$$R_{jl}-\frac{1}{2}g_{jl}R+g_{jl}\lambda=T_{jl} \tag{47}$$

Equation (47) represents the Einstein field equation incorporating the cosmological constant, which arises as an integration constant during the preceding integration process.

Alternatively, the identical outcome in Equation (47) can be obtained by employing Equations (40) and (41) in the following manner.

$$\frac{1}{n}g_{jl}R=g_{jl}\lambda-\frac{n-2}{2n}g_{jl}\tilde{T}=g_{jl}\lambda-\frac{1}{n}g_{jl}T \tag{48}$$

Upon applying Equation (48), a remarkable correlation between the scalars in general relativity and the cosmological constant can be derived as the following,

$$R=n\lambda-\frac{n-2}{2}\tilde{T} \tag{49}$$

$$R=n\lambda-T \tag{50}$$

The result of the integration in Equation (45) remains unaltered in Equation (50). In Equation (49), it becomes apparent that the cosmological constant is connected to the curvature of space resulting from the presence of vacuum energy. When the scalar curvature R reaches zero, it is observed that the cosmological constant is directly proportional to the interaction energy of space, which contains comprehensive spatial information.

In simpler terms, the value of the cosmological constant is determined based on the state of space and its interaction with the energy present within it. Therefore, it is necessary to examine the initial conditions of the entire system in order to define the cosmological constant λ. This definition is constrained by compatible limitations on the initial data, rather than being a fixed value governed by universal physical laws. This concept was previously referred to as the constant of integration in Equation (45) [19] [20].

When λ is equal to zero, Equation (50) transforms into a trace reverse ($R=-T$), resembling the Einstein field equation in four dimensions. Through these discoveries, we have gained insights into the characteristics of the cosmological constant, which elucidate how the curvature of space aligns with the energy requirements that interact with it. This interaction is manifested in the form of vacuum energy, as demonstrated in Equations (49) and (50) [21]. The effective cosmological constant Λ serves as a constant of integration in classical GR, and thus can be arbitrarily chosen. Consequently, it is not reliant on any fundamental value assigned to Λ. Therefore, utilizing the trace-free Einstein (TFE) equations instead of the Einstein field equations (EFE) appears to be a sound theoretical assumption. In this scenario, a large vacuum cosmological constant Λ_{vac} has no influence on cosmology or the solar system, as the zero-point energy does not impact the geometry of space-time. The EFE remains unchanged, except for the inclusion of Λ as an integration constant, which can assume a small or zero value. Empirical observations indicate that this constant corresponds to a specific cosmological length scale ($\Lambda = H_0^2$), which should be determined based on the initial conditions of our universe. Consequently, the vacuum energy problem is resolved within the framework of trace-free Einstein gravity, while the coincidence problem, characterized by the near equality between Λ and the Hubble constant, persists [20].

The experimental forecasts of both theories exhibit identical outcomes, thereby rendering experiments incapable of distinguishing between them, save for a fundamental characteristic. This characteristic pertains to the Einstein Field Equation (EFE), which has been verified with great precision through solar system observations and binary pulsar measurements. However, when considering the prediction for the vacuum energy density based on Quantum Field Theory, the EFE yields an incorrect result by several orders of magnitude. Conversely, the Theory of Fundamental Energy (TFE) does not encounter this discrepancy. Consequently, experimental evidence strongly favors the TFE in this regard [19] [20].

5. Conclusions

The hypothesis put forth in this study integrates the duality principle between space and energy in order to provide a comprehensive explanation of Einstein's field equation. A model has been developed to shed light on the complexities of information transmission between space and energy. This model suggests that an intermediary is essential to facilitate the efficient transfer of information between these two entities. By adopting this innovative approach, the resulting equations not only adhere to the Einstein field's equation but also the Trace-free Einstein's field equation, eliminating the need for intricate integrations. Instead, a suitable equivalent Riemannian tensor is introduced to formulate the terms of interaction.

Equation (11) presents a more extensive range of applicability as it serves as a generalization of Einstein's 4-index equation. This equation encompasses a broader spectrum of scenarios, allowing for a more comprehensive understanding of the underlying phenomena. The vacuum components within this equation exhibit similarities to those obtained through the utilization of the principle of least action.

A novel 4-index gravitational field equation has been proposed, which incorporates Weyl's tensor and Riemann curvature tensor in a linear manner. This equation includes the energy-momentum tensor of the actual gravitational field, which is absent in Einstein's field equation due to the contraction of the four-index equation resulting in the loss of Weyl's tensor. The newly derived formula represents a natural extension of the well-known Einstein's equation with a two-index and is limited to the domain of general relativity. The presence of Weyl's tensor in this equation provides additional information, supporting the adoption of a fourth-order theory. Equation (15) demonstrates the utilization of internal energy (vacuum energy) through the deformation of empty space, while Equation (17) clarifies that the transmission of information regarding the desired curvature is facilitated by matter. However, the generation of this spatial configuration necessitates the utilization of energy resources. Hence, equations (15) and (17) depict two aspects of the dual energy space process.

In the context of Einstein's field equation and the Trace-free Einstein's field equation, the constant has been associated with the mechanism for storing space and energy information, facilitating their mutual recognition during gravitational interaction.

It is noteworthy that Equations (49) and (50) exhibit a significant correlation with the relation presented in (48), which is derived from Equation (47). This correlation reveals a connection between the cosmological constant and the curvature of space that arises due to the presence of vacuum energy. Upon reaching zero scalar curvature R, it becomes evident that the cosmological constant is directly proportional to the vacuum energy of space, which encompasses inclusive spatial information.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Landau, L. and Lifchitz, E. (1975) The Classical Theory of Fields. Pergamon Oxford Press, New York.

[2] Misner, C.W., Thorne, K.S. and Wheeler, J.A. (1973) Gravitation. Freeman W.H., New York.

[3] Weinberg, S. (1972) Gravitation and Cosmology. Wiley, New York.

[4] Eisenhart, L.P. (1926) Riemannian Geometry. Princeton University Press, Princeton.

[5] Carroll, S. (2004) Spacetime and Geometry: An Introduction to General Relativity. Addison-Wesley, San Francisco.

[6] Blau, M. (2014) Lecture Notes on General Relativity. Albert Einstein Center for Fundamental Physics, Bern.

[7] Zakir, Z. (1999) New Equations for the Gravitation with Riemann Tensor and 4-Index Energy-Momentum Tensors for the Gravitation and the Matter. arXiv: gr-qc/9905009.

[8] Zakir, Z. (1999) Four-Index Equations for the Gravitation and the Gravitational Energy-Momentum Tensor. arXiv: gr-qc/9906039.

[9] Moulin, F. (2017) Generalization of Einstein's Gravitational Field Equations. *The European Physical Journal C*, **77**, Article No. 878. https://doi.org/10.1140/epjc/s10052-017-5452-y

[10] Gribbin, J. and Gribbin, M. (1997) Richard Feynman—A Life in Science. Penguin Books, New York.

[11] Puthoff, H.E. (1989) Gravity as a Zero-Point-Fluctuation Force. *Physical Review A*, **39**, 2333-2342. https://doi.org/10.1103/PhysRevA.39.2333

[12] Colella, R., Overhauser, A.W. and Werner, S.A. (1975) Observation of Gravitationally Induced Quantum Interference. *Physical Review Letters*, **34**, 1472-1474. https://doi.org/10.1103/PhysRevLett.34.1472

[13] Sakharov, A.D. (1968) Vacuum Quantum Fluctuations in Curved Space and the Theory of Gravitation. *Soviet Physics—Doklady*, **12**, 1040-1041.

[14] Málek, T. (2012) General Relativity in Higher Dimensions. arXiv: 1204.0291v1.

[15] Stelle, K.S. (1978) Classical Gravity with Higher Derivatives. *General Relativity and Gravitation*, **9**, 353-371. https://doi.org/10.1007/BF00760427

[16] Aubin, T. (1998) Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag, Berlin. https://doi.org/10.1007/978-3-662-13006-3

[17] Gallot, S., Hulin, D. and Lafontaine, J. (2004) Riemannian Geometry. 3rd Edition, Springer-Verlag, Berlin. https://doi.org/10.1007/978-3-642-18855-8

[18] Petersen, P. (2016) Riemannian Geometry. Springer, Cham. https://doi.org/10.1007/978-3-319-26654-1

[19] Ellis, G.F.R. (2012) The Trace-Free Einstein Equations and Inflation. arXiv: 1306.3021v3.

[20] Ellis, G.F.R., van Elst, H., Murugan, J. and Uzan, J.P. (2011) On the Trace-Free Einstein Equations as a Viable Alternative to General Relativity. *Classical and Quantum Gravity*, **28**, Article ID: 225007. https://doi.org/10.1088/0264-9381/28/22/225007

[21] Weinberg, S. (1989) The Cosmological Constant Problem. *Reviews of Modern Physics*, **61**. https://doi.org/10.1103/RevModPhys.61.1

Appendixes

Appendix 1

The specific fourth-order tensors, which are frequently cited in prominent literature [1]-[6], are the metric tensor combinations that will be utilized in our computations.

$$T_{ijkl}+\beta\tilde{T}_{ijkl}=\beta_1 g_{ijkl}\tilde{T}+\beta_2\left(g_{ijk\rho}\tilde{T}_l^{\rho}+g_{ij\rho l}\tilde{T}_k^{\rho}\right) \tag{A1.1}$$

$$R_{ijkl}+\alpha\tilde{R}_{ijkl}=\alpha_1 g_{ijkl}\tilde{R}+\alpha_2\left(g_{ijk\rho}\tilde{R}_l^{\rho}+g_{ij\rho l}\tilde{R}_k^{\rho}\right) \tag{A1.2}$$

In a n-dimensional space, $\delta_\sigma^l=n$ if $l=\sigma$. To find the parameters in equations (A1.1) and (A1.2), we contract them in 4 indices.

$$g^{ik}g^{jl}T_{ijkl}=\grave{\beta}_1 g^{ik}g^{jl}g_{ijkl}\tilde{T}+\grave{\beta}_2 g^{ik}g^{jl}\left(g_{ijk\rho}\tilde{T}_l^{\rho}+g_{ij\rho l}\tilde{T}_k^{\rho}\right) \tag{A1.3}$$

$$g^{ik}g^{jl}R_{ijkl}=\grave{\alpha}_1 g^{ik}g^{jl}g_{ijkl}\tilde{R}+\grave{\alpha}_2 g^{ik}g^{jl}\left(g_{ijk\rho}\tilde{R}_l^{\rho}+g_{ij\rho l}\tilde{R}_k^{\rho}\right) \tag{A1.4}$$

The contraction of the metric gives,

$$g^{ik}g^{jl}R_{ijkl}=R;\quad g^{ik}g^{jl}T_{ijkl}=T;\quad g^{ik}g^{jl}g_{ijkl}=n(n-1);$$

$$g^{jl}g_{ijkl}=(n-1)g_{ik};\quad g^{ik}g_{ik}=n$$

$$g^{ik}\left(g_{ijk\rho}\tilde{R}_l^{\rho}+g_{ij\rho l}\tilde{R}_k^{\rho}\right)=(n-2)\tilde{R}_{jl}+g_{jl}\tilde{R}$$

$$\begin{aligned} g^{ik}g^{jl}\left(g_{ijk\rho}\tilde{T}_l^{\rho}+g_{ij\rho l}\tilde{T}_k^{\rho}\right) &= g^{jl}\left(g^{ik}g_{ijk\rho}\right)\tilde{T}_l^{\rho}+g^{ik}\left(g^{jl}g_{ij\rho l}\right)\tilde{T}_k^{\rho} \\ &= g^{jl}(n-1)g_{j\rho}\tilde{T}_l^{\rho}+g^{ik}(n-1)g_{i\rho}\tilde{T}_k^{\rho} \\ &= (n-1)g^{jl}\tilde{T}_{jl}+(n-1)g^{ik}\tilde{T}_{ik} \\ &= 2(n-1)\tilde{T} \end{aligned}$$

Then we find,

$$g^{ik}g^{jl}\left(g_{ijk\rho}\tilde{R}_l^{\rho}+g_{ij\rho l}\tilde{R}_k^{\rho}\right)=2(n-1)\tilde{R} \tag{A1.5}$$

$$g^{ik}g^{jl}g_{ijkl}\tilde{T}=(n-1)g^{ik}g_{ik}\tilde{T}=n(n-1)\tilde{T} \tag{A1.6}$$

$$g^{ik}g^{jl}g_{ijkl}\tilde{R}=n(n-1)\tilde{R} \tag{A1.7}$$

The contraction of Equation (A1.1) and (A1.2) gives,

$$T+\beta\tilde{T}=\beta_2 2(n-1)\tilde{T}+\beta_1 n(n-1)\tilde{T} \tag{A1.8}$$

$$R+\alpha\tilde{R}=\alpha_2 2(n-1)\tilde{R}+\alpha_1 n(n-1)\tilde{R} \tag{A1.9}$$

$$1+\beta=\beta_2 2(n-1)+\beta_1 n(n-1) \tag{A1.10}$$

$$1+\alpha=\alpha_2 2(n-1)+\alpha_1 n(n-1) \tag{A1.11}$$

The factor of $R = 1$ and we assume at the boundaries, $T=\tilde{T}$, $R=\tilde{R}$

Finally, we find the parameters

$$\alpha_2=\frac{1-\alpha}{n-2};\quad \alpha_1=-\frac{1-\alpha}{(n-1)(n-2)};\quad \beta_2=\frac{1-\beta}{n-2};\quad \beta_1=-\frac{1-\beta}{(n-1)(n-2)} \tag{A1.12}$$

Substitute (A1.12) in Equation (8) we find

$$\alpha R_{ijkl} - \beta T_{ijkl} + \frac{1-\alpha}{n-2}\left(g_{ijk\rho}\tilde{R}_l^{\rho} + g_{ij\rho l}\tilde{R}_k^{\rho}\right) + \frac{1-\beta}{n-2}\left(g_{ijk\rho}\tilde{T}_l^{\rho} + g_{ij\rho l}\tilde{T}_k^{\rho}\right)$$
$$-\frac{1-\alpha}{(n-1)(n-2)} g_{ijkl}\tilde{R} + \frac{1-\beta}{(n-1)(n-2)} g_{ijkl}\tilde{T} = 0$$

Appendix 2

$$\begin{aligned}&\alpha R_{ijkl} + \frac{1-\alpha}{n-2}\left(g_{ijk\rho}\tilde{R}_l^{\rho} + g_{ij\rho l}\tilde{R}_k^{\rho}\right) - \frac{n-2\alpha}{2(n-1)(n-2)} g_{ijkl}\tilde{R}\\ &= \varkappa\left\{\frac{2}{n-2}\left(g_{ijk\rho}\tilde{T}_l^{\rho} + g_{ij\rho l}\tilde{T}_k^{\rho}\right) - \frac{1}{2(n-1)(n-2)} g_{ijkl}\tilde{T} + 2T_{ijkl}\right\} + T_{ijkl}\end{aligned} \tag{A2.1}$$

$$R_{ijkl} - \frac{1}{n-2}\left(g_{ijk\rho}\tilde{R}_l^{\rho} + g_{ij\rho l}\tilde{R}_k^{\rho}\right) + \frac{1}{(n-1)(n-2)} g_{ijkl}\tilde{R} = \alpha^{-1}T_{ijkl} \tag{A2.2}$$

$$R_{ijkl} = \alpha^{-1}T_{ijkl} + \frac{1}{n-2}\left(g_{ijk\rho}\tilde{R}_l^{\rho} + g_{ij\rho l}\tilde{R}_k^{\rho}\right) - \frac{1}{(n-1)(n-2)} g_{ijkl}\tilde{R} \tag{A2.3}$$

$$\begin{aligned}&\frac{1}{n-2}\left(g_{ijk\rho}\tilde{R}_l^{\rho} + g_{ij\rho l}\tilde{R}_k^{\rho}\right) - \frac{n}{2(n-1)(n-2)} g_{ijkl}\tilde{R}\\ &= \varkappa\left\{\frac{2}{n-2}\left(g_{ijk\rho}\tilde{T}_l^{\rho} + g_{ij\rho l}\tilde{T}_k^{\rho}\right) - \frac{1}{2(n-1)(n-2)} g_{ijkl}\tilde{T} + 2T_{ijkl}\right\}\end{aligned} \tag{A2.4}$$

$$G_{jl} = \tilde{R}_{jl} + \frac{1}{n-2} g_{jl}\tilde{R} - \frac{n}{2(n-2)} g_{jl}\tilde{R} \tag{A2.5}$$

$$g^{ik}g^{jl}\left(g_{ijk\rho}\tilde{R}_l^{\rho} + g_{ij\rho l}\tilde{R}_k^{\rho}\right) = 2(n-1)\tilde{R}, \quad g^{ik}g^{jl}g_{ijkl}\tilde{R} = n(n-1)\tilde{R},$$

$$G = \left\{\frac{2(n-1)}{n-2} - \frac{n^2}{2(n-2)}\right\}\tilde{R} \tag{A2.6}$$

$$\frac{1}{2\alpha\varkappa} g^{ik}G_{ijkl} = g^{ik}\mathcal{C}_{ijkl} + \frac{1}{\alpha} g^{ik}\mathcal{T}_{ijkl} + g^{ik}\left\{R_{ijkl} - \tilde{R}_{ijkl}\right\} \tag{A2.7}$$

Appendix 3

$$\Lambda = \frac{n-2}{2n}\left\{\tilde{R} - \tilde{T}\right\}$$

$$S = \frac{1}{2}\int (R - 2\Lambda)\sqrt{-g}\,\mathrm{d}^4x$$

Taking variations with respect to the inverse metric:

$$\delta S = 0 = \delta\left[\frac{1}{2}\int (R - 2\Lambda)\sqrt{-g}\,\mathrm{d}^4x\right]$$

$$\delta S = \int\left[\frac{\sqrt{-g}}{2}\frac{\delta R}{\delta g^{ik}} + \frac{R}{2}\frac{\delta\sqrt{-g}}{\delta g^{ik}} - \Lambda\frac{\delta\sqrt{-g}}{\delta g^{ik}}\right]\delta g^{ik}\,\mathrm{d}^4x$$

$$\delta S = \int\left[\frac{1}{2}\frac{\delta R}{\delta g^{ik}} + \frac{R}{2\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{ik}} - \frac{\Lambda}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{ik}}\right]\delta g^{ik}\sqrt{-g}\,\mathrm{d}^4x = 0$$

$$\frac{\delta R}{\delta g^{ik}} = R_{ik}, \quad \frac{R}{2\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{ik}} = -\frac{R}{4}g_{ik}, \quad \frac{\Lambda}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{ik}} = -\frac{\Lambda}{2}g_{ik}$$

$$R_{ik} - \frac{1}{2} R g_{ik} + \Lambda g_{ik} = 0$$

Because the interaction terms in vacuum remains constants, $\left\{ \tilde{R} - \tilde{T} \right\}$ is a constant.